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The Goryachev–Chaplygin gyrostat in quantum mechanics

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Abstract. We investigate the quantum mechanical analogue of the classical integrable system named in the title. The Goryachev–Chaplygin (GC) gyrostat is a generalisation of the GC top, where the Coriolis interaction is taken into account. The problem is formulated in terms of the Euclid E(3) group. The integrals of motion are derived. The separation of variables is based on the connection between the degenerated representation of the E(3) group and the special representation of the SO(3, 2) group. Some numerical results on the spectra of energy and separation constant are presented. The strong field limit for the integrals of motion is considered in detail, as well as a correlation diagram connecting the states in the limits of weak and strong fields. It appears that the dependence of energy and separation constant on the strength of the Coriolis interaction has the zone behaviour.

1. Introduction

The quantum analogue of the integrable classical system found by Goryachev and Chaplygin (GC) in 1900 has been investigated in the series of papers by Komarov (1982a, b, c). The system is defined as a top in external constant homogeneous field, and the principal momenta relate as $1:1:\frac{1}{4}$ and the dipole moment of the top is directed across its symmetry axis. For the system to be integrable it is necessary that the projection of angular momentum on the field is zero.

In the present paper we analyse a generalisation of the GC top—the Goryachev–Chaplygin gyrostat (GCG). The Hamiltonian of the gyrostat differs from that of the corresponding top by linear terms in the body-frame:

$$H_g = H_t + (\boldsymbol{J}, \boldsymbol{\lambda}), \tag{1}$$

where $\boldsymbol{J}$ is an angular momentum and $\boldsymbol{\lambda}$ is a constant arbitrary vector, the gyrostatic moment. A physical realisation of a gyrostat in classical mechanics is, for instance, the body with fluid-filled cavities. As the body rotates, the liquid moves in the cavities and produces additional angular momentum (Gorr *et al* 1978). A model Hamiltonian of the gyrostat is used also for the description of the interaction of rotational and vibrational degrees of freedom in molecules due to the Coriolis interaction (Herzberg 1945).

The structure of the paper is similar to that of Komarov (1982a, b, c). In § 2 the integrals of motion are given in terms of the Euclid group E(3) and the procedure of separation of variables is outlined. In § 3 the spectral problem for energy and separation constant operators is solved. In § 4 we study the strong field limit for the integrals of motion. Finally in § 5 classification of GCG states and the correlation diagram for the energy levels are discussed in detail.

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2. Integrals of motion for GCG and separation of variables

For the GC gyrostat the principal momenta of inertia are in the ratio $1:1:\frac{1}{4}$, and the gyrostatic moment is directed along its symmetry axis $\boldsymbol{\lambda} = (0, 0, \lambda)$. The components of the angular momentum J_i and the unit field vector $\boldsymbol{n} = (x_1, x_2, x_3)$ in the body-frame satisfy the left commutation relations of the Euclid algebra e(3)

$$\begin{aligned} &[J_i, J_k] = -\mathrm{i}\varepsilon_{ikl}J_l, \\ &[J_i, x_k] = -\mathrm{i}\varepsilon_{ikl}x_l, \\ &[x_i, x_k] = 0, \qquad i, k, l = 1, 2, 3. \end{aligned} \tag{2}$$

The Hamiltonian GCG

$$H_\lambda = \tfrac{1}{2}(J_1^2 + J_2^2 + 4(J_3 - \lambda)^2) - bx_1 \tag{3}$$

commutes with the separation constant operator

$$G_\lambda = 2(J_3 - 2\lambda)(J_1^2 + J_2^2 + \tfrac{1}{4}) + b\{x_3, J_1\} \tag{4}$$

provided that the projection of the angular momentum onto the field direction is zero,

$$l = x_1J_1 + x_2J_2 + x_3J_3 = 0. \tag{5}$$

Here l and $x_ix_i \equiv r^2 = 1$ are Casimir operators of the E(3) group. Thus for the GCG to be integrable it is necessary to have the special ratio of principal momenta of inertia and the zero value of the Casimir operator l, while the vectors of the dipole and gyrostatic moments are directed perpendicular to and along its symmetry axis respectively.

The separation of variables in the spectral problem for the operators H_λ and G_λ is done in the same manner as for the GC top. It is based on the connection of the degenerated representation of the the E(3) group with a special representation of the SO(3, 2) group. The ten generators of the latter are

$$\begin{aligned} &J_i = -\varepsilon_{ikl}G_{kl}, \\ &P_i = \mathrm{i}G_{i4}, \qquad R_i = \mathrm{i}G_{i5}, \qquad S = G_{45}. \end{aligned} \tag{6}$$

They satisfy the commutation relations

$$[G_{IJ}, G_{KL}] = \mathrm{i}(\delta_{IL}G_{KJ} + \delta_{IK}G_{JL} + \delta_{JK}G_{LI} + \delta_{JL}G_{IK}) \tag{7}$$

and additional identities

$$\varepsilon_{IPQRS}G_{PQ}G_{RS} = 0, \qquad G_{IS}G_{SJ} + G_{JS}G_{SI} = \delta_{IJ}. \tag{8a, b}$$

The algebra (6)–(8) usually arises in connection with the Majorana representation of the Lorentz group (Barut and Rączka 1977). The generators G_{IS} could be represented by the bilinear forms of two types of boson creation–annihilation operators. It is easy to check that J_i taken from (6) and x_i defined as

$$x_i = S^{-1/2}R_iS^{-1/2} \tag{9}$$

are the generators of the E(3) group and $l = 0$, $x_ix_i = 1$.

The group SO(3, 2) has a semisimple subgroup SO(2, 2) with generators $Q_{ab}(a, b = 1, 2, 3, 4)$, which can be obtained by excluding the third column and third row from

the 5×5 skew-symmetric matrix G_{IJ},

$$Q_{12}=-J_3, \qquad Q_{13}=P_1, \qquad Q_{23}=P_2, \qquad Q_{34}=S, \qquad Q_{14}=R_1, \qquad Q_{24}=R_2. \tag{10}$$

The SO(2, 2) group is a dynamical group for the GCG. The separation of variables is connected with a decomposition of the SO(2, 2) group into the direct product $SO(2,1)\times SO(2,1)$ with subgroup generators

$$M_i^{(\alpha)}=\tfrac{1}{2}(-\varepsilon_{ikl}Q_{kl}\pm Q_{i4}). \tag{11}$$

Here minus corresponds to $\alpha=1$ and plus to $\alpha=2$. $M_i^{(\alpha)}$ satisfy the commutation relations

$$\begin{aligned}&[M_i^{(\alpha)}, M_k^{(\beta)}]=\mathrm{i}\delta^{\alpha\beta}g_{is}\varepsilon_{skl}M_l^{(\alpha)},\\ &g_{11}=g_{22}=-g_{33}=1\end{aligned} \tag{12}$$

and the Casimir operators in the chosen representation are

$$C=M_1^{(\alpha)2}+M_2^{(\alpha)2}-M_3^{(\alpha)2}=\tfrac{3}{16}. \tag{13}$$

To separate variables let us write down the Hamiltonian of GCG in Q_{ab} variables

$$H_\lambda=\tfrac{1}{2}(S^2-\tfrac{1}{4}+3J_3^2-8\lambda J_3+4\lambda^2)-bS^{-1/2}R_1S^{-1/2} \tag{14}$$

and pass to the energy representation

$$H_\lambda|E, g\rangle=E|E, g\rangle. \tag{15}$$

If we multiply the equation (14) from the left by $S^{1/2}$ and use new basis functions $S^{-1/2}|E, g\rangle$, the equations more naturally can be written in terms of generators $M_i^{(\alpha)}$. It decomposes into the two equations

$$G_\lambda^{(1)}|g_k^{(1)}\rangle=g_k^{(1)}|g_k^{(1)}\rangle, \tag{16}$$

$$G_\lambda^{(2)}|g_q^{(2)}\rangle=g_q^{(2)}|g_q^{(2)}\rangle, \tag{17}$$

where $S^{-1/2}|E, g\rangle=|g_k^{(1)}\rangle|g_q^{(2)}\rangle$ and

$$G_\lambda^{(\alpha)}(E, \lambda, b)=(-1)^\alpha[8M_3^{(\alpha)3}-16\lambda M_3^{(\alpha)2}-4(E+\tfrac{1}{8}-2\lambda^2)M_3^{(\alpha)}]-4bM_1^{(\alpha)}. \tag{18}$$

Here $\alpha=1, 2$ and k, q label the eigenstates. The energy and separation constant as functions of b, k, q are obtained from the transcendental equation

$$g_k^{(1)}=g_q^{(2)}=g_{kq}(\lambda, b). \tag{19}$$

In the new variables $u=2M_3^{(2)}$ and $v=-2M_3^{(1)}$ the quantum equations of motion for GCG look like their classical counterparts (Sretensky 1953)

$$\frac{\mathrm{d}}{\mathrm{d}t}u=(u-v)^{-1/2}[4b^2(u^2+\tfrac{3}{4})-U^2(u)]^{1/2}(u-v)^{-1/2}, \tag{20a}$$

$$\frac{\mathrm{d}}{\mathrm{d}t}v=(v-u)^{-1/2}[4b^2(v^2+\tfrac{3}{4})-U^2(v)]^{1/2}(v-u)^{-1/2}, \tag{20b}$$

$$U(\mu)=\mu^3-4\lambda\mu^2-2(E+\tfrac{1}{8}-2\lambda^2)\mu-G_\lambda^{(\alpha)}, \tag{21}$$

where $U(\mu)$ is a cubic operational form.

In the papers by Komarov (1982a, b, c) it was pointed out that the GC top is similar to the three-body periodic Toda lattice, if the impulse of the centre of mass of the lattice is equal to zero. This analogy is true also for the GC gyrostat. When one replaces in the equations (20) $4b^2(\mu^2+\frac{3}{4})$ by $4b^2$ in the corresponding cubic form (21), the GCG equations in classical mechanics coincide with the equations of motion of the three-body periodic Toda lattice, if the impulse of the centre of mass equals 4λ. Thus the coefficient 4λ before μ^2 in $U(\mu)$ has the same meaning as the impulse of the lattice centre of mass.

3. Determination of the spectra of the integrals of motion

The eigenfunctions of the operators $G^{(\alpha)}(E, \lambda, b)$ can be constructed in the basis of the eigenfunctions $|n_\alpha\rangle$ of the compact generators $M_3^{(\alpha)}$. It is natural to label the GCG state by quantum numbers L and M that define the angular momentum and its third component respectively in absence of the field, i.e. when $b=0$.

Let us define $n_\alpha = 2l+\delta$, where δ equals 0 or 1,

$$\delta = \tfrac{1}{2}(1-(-1)^{L+M}). \tag{22}$$

The expansion for the eigenvectors with the fixed parity can be written as

$$|g^{(\alpha)}\rangle = \sum_{l=0}^{\infty} A_l^{(\alpha)}(-1)^{\alpha l}|n_\alpha\rangle. \tag{23}$$

For the coefficients $A_l^{(\alpha)}$ one gets the three-term recursion relations (TTRR)

$$\begin{aligned} &A_{l+1}^{(\alpha)}\gamma_l^{(\alpha)} + A_l^{(\alpha)}\beta_l^{(\alpha)} + A_{l-1}^{(\alpha)}\gamma_{l-1}^{(\alpha)} = 0, \qquad A_{-1}^{(\alpha)} = 0, \\ &\gamma_l^{(\alpha)} = -b[(n_\alpha+1)(n_\alpha+2)]^{1/2}, \\ &\beta_l^{(\alpha)} = (n_\alpha+\tfrac{1}{2})^3 - (-1)^\alpha 4\lambda(n_\alpha+\tfrac{1}{2})^2 - 2(E+\tfrac{1}{8}-2\lambda^2)(n_\alpha+\tfrac{1}{2}) - (-1)^\alpha g^{(\alpha)}. \end{aligned} \tag{24}$$

The TTRR (24) differ from those of the GC top by the quadratic term in $\beta_l^{(\alpha)}$. The roots of the determinant of the TTRR matrix (24) can be calculated with the help of infinite continuous fractions that rapidly converge due to the cubic increase of $\beta_l^{(\alpha)}$ as $l \to \infty$.

From the analysis of the TTRR (24) the following symmetry can be established

$$\begin{aligned} &E(L, M, \lambda, b) = E(L, -M, -\lambda, b), \\ &g(L, M, \lambda, b) = -g(L, -M, -\lambda, b). \end{aligned} \tag{25}$$

The algorithm for the calculation of E and g is practically the same as for the GC top (Komarov 1982b). The small field asymptotic gives a good starting approximation for linear extrapolation of the first step. In second-order perturbation theory one has

$$\begin{aligned} E &= \tfrac{1}{2}[L(L+1)+3M^2-8\lambda M+4\lambda^2] - \frac{b^2}{2(2L+1)}(\theta(L, M, \lambda)+\theta(L, -M, -\lambda)) + O(b^4), \\ g &= 2(M-2\lambda)(L-M+\tfrac{1}{2})(L+M+\tfrac{1}{2}) \\ &\quad - \frac{b^2}{2L+1}[(L-M+\tfrac{1}{2})\theta(L, M, \lambda) - (L+M+\tfrac{1}{2})\theta(L, -M, -\lambda)] + O(b^4), \end{aligned} \tag{26}$$

where

$$\theta(L, M, \lambda) = \frac{(L+M+1)(L+M+2)}{(2L+3)(2L+6M+5-8\lambda)} - \frac{(L+M-1)(L+M)}{(2L-1)(2L+6M-3-8\lambda)}.$$

For the next steps, we used a quadratic extrapolation of $E(b)$, $g(b)$ and its derivatives at a fixed λ. Then the calculations were made at each point of b with a required accuracy by iterations in the continuous fractions of TTRR (24).

Some numerical results are given in figures 1 and 2 for the two values of λ. The states are labelled by (L, M). As well as the GC top, the GC gyrostat demonstrates the two patterns of behaviour of $E(b)$. In the first case the energy decreases monotonously, while b increases. The GCG at once aligns its dipole moment along the field. In the second case the energy has a maximum. The gyrostat at first untwists, increasing its energy, and only then draws in the field.

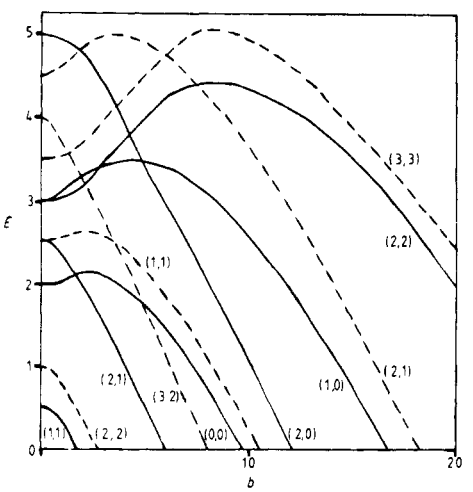

Figure 1. The eigenenergies of GCG as a function of the field strength for $\lambda = 1.0$ (full curves) and $\lambda = 2.0$ (broken curves). Each curve is labelled by the quantum numbers (L, M).

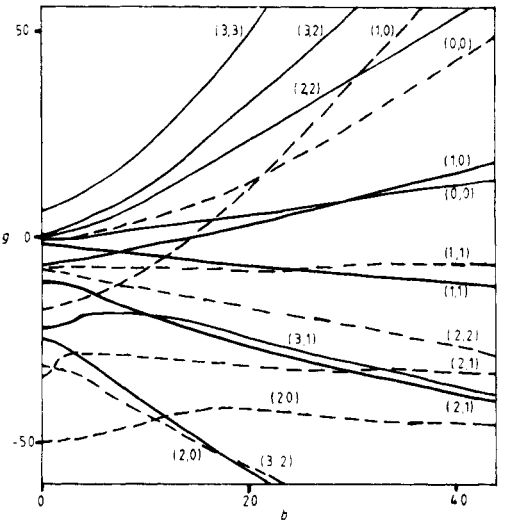

Figure 2. The eigenvalues of the separation constant of GCG as a function of field strength for $\lambda = 1.0$ (full curves) and $\lambda = 2.0$ (broken curves). Each curve is labelled by the quantum numbers (L, M).

4. Strong field limit

In a strong field the dipole moment of the gyrostat is directed along the field to minimise its energy, $E \sim -b$, $x_1 \sim 1$. Thus it is useful to consider the scale transformation

$$\begin{aligned} x_1 &= \xi_1, & x_2 &= \varkappa\xi_2, & x_3 &= \varkappa\xi_3, \\ p_1 &= \pi_1, & p_2 &= \varkappa^{-1}\pi_2, & p_3 &= \varkappa^{-1}\pi_3, \end{aligned} \tag{27}$$

with $\varkappa = b^{-1/4}$ as a scale factor. Here x_i, p_i are the components of coordinate and impulse of the gyrostat point with the Poisson vector $\boldsymbol{n} = (0, 0, 1)$ in the body-frame.

We get from (27) and the Casimir operators $l = x_i J_i = 0$, $x_i x_i = 1$ the following expansions

$$\begin{aligned} &\xi_1 = 1 - \tfrac{1}{2}\varkappa^2(\xi_2^2 + \xi_3^2) + O(\varkappa^4), \\ &J_1 = O(1), \qquad J_2 = \varkappa^{-1}\pi_3 + O(\varkappa), \qquad J_3 = -\varkappa^{-1}\pi_2 + O(\varkappa). \end{aligned} \tag{28}$$

For the GCG Hamiltonian we have

$$H_\lambda = -b + b^{1/2}H_1 + \ldots, \qquad E = -b + b^{1/2}\varepsilon_1 + O(1), \tag{29}$$

where

$$H_1 = \tfrac{1}{2}(\pi_3^2 + 4(\pi_2 - \gamma)^2 + \xi_2^2 + \xi_3^2), \qquad \gamma = \lambda\varkappa = O(1).$$

The new Hamiltonian corresponds to the two-dimensional oscillator with the frequencies ratio 1:2 and the shifted on π_2 centre of equilibrium. The spectrum of H_1 is degenerated. The eigenvectors must be classified with respect to the separation constant operator, which after the scale transformation takes the form

$$\begin{aligned} &G_\lambda = b^{3/4}G_0 + \ldots, \qquad g = g_0 b^{3/4} + O(b^{1/4}), \\ &G_0 = \{\xi_3, \xi_3\pi_2 - \xi_2\pi_3\} - 2\pi_2\pi_3^2 - 4\gamma\pi_3^2. \end{aligned} \tag{30}$$

It is useful to introduce the creation–annihilation operators for the two one-dimensional oscillators

$$\begin{aligned} &\xi_2 = c_2 + c_2^+, \qquad \pi_2 - \gamma = (c_2 - c_2^+)/2\mathrm{i}, \\ &\xi_3 = (c_3 + c_3^+)2^{-1/2}, \qquad \pi_3 = -\mathrm{i}(c_3 - c_3^+)2^{-1/2}, \end{aligned} \tag{31}$$

which obey the usual commutation relations

$$[c_i, c_k^+] = \delta_{ik}, \qquad [c_i, c_k] = 0, \qquad i, k = 2, 3. \tag{32}$$

Then we can rewrite the limiting integrals of motion in the form

$$\begin{aligned} &H_1 = \tfrac{1}{2}(\{c_3^+, c_3\} + 2\{c_2^+, c_2\}), \\ &G_0 = -2\mathrm{i}(c_2 c_3^{+2} - c_2^+ c_3^2) - 2\gamma\{c_3^+, c_3\}, \end{aligned} \tag{33}$$

where $\{\,,\}$ is an anticommutator. In the basis of eigenfunctions of these oscillators one can get the spectrum of H_1 and G_0. We have

$$\varepsilon_1 = m_3 + 2m_2 + \tfrac{3}{2} = N + \tfrac{3}{2} \tag{34}$$

and g_0 is an eigenvalue of the tridiagonal Hermitian matrix

$$\begin{aligned} &\langle m_2 - 1, m_3 + 2|G_0|m_2, m_3\rangle = -2\mathrm{i}[m_2(m_3+1)(m_3+2)]^{1/2}, \\ &\langle m_2 + 1, m_3 - 2|G_0|m_2, m_3\rangle = 2\mathrm{i}[(m_2+1)m_3(m_3+1)]^{1/2}, \\ &\langle m_2, m_3|G_0|m_2, m_3\rangle = -2\gamma(2m_3+1). \end{aligned} \tag{35}$$

Figure 3 shows some examples of dependence of eigenvalues g_0 of the matrix (35) on $\gamma = \lambda\varkappa$ and on the index $s = k + q$.

The same results can be obtained by the separation of variables in the equation $H_1|\psi\rangle = \varepsilon_1|\psi\rangle$. Let us introduce new variables $\mu = -\pi_2 + (\pi_2^2 + \pi_3^2)^{1/2}$ and $\nu = -\pi_2 - (\pi_2^2 + \pi_3^2)^{1/2}$. They are the parabolic coordinates in the two-dimensional impulse space. Supposing that the eigenfunctions of H_1 can be represented as a product $|\psi\rangle = B(\mu)B(\nu)$

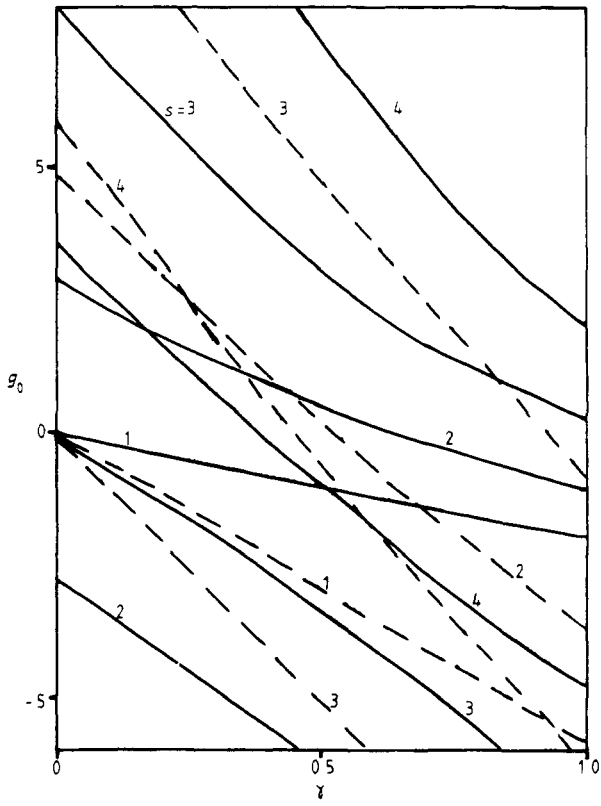

Figure 3. The spectrum of the separation constant g_0 in the strong field limit as a function of $\gamma = \lambda\varkappa$ and $s = k + q$, $s = 1, 2, 3, 4$. The full curves correspond to even states and the broken curves correspond to odd ones.

one gets two differential equations of the type

$$\frac{\partial}{\partial\mu}\mu\frac{\partial}{\partial\mu}B(\mu) - \tfrac{1}{4}\left(\mu^3 + 4\gamma\mu^2 - (2\varepsilon_1 - 4\gamma)\mu - g_0(-1)^\alpha + \frac{1}{4\mu}\right)B(\mu) = 0. \tag{36}$$

The variable μ is positive for $\alpha = 2$, and negative for $\alpha = 1$. After the substitution, which extracts singular points at zero and infinity, the solutions of equations (36) can be regarded as series in powers of μ. The coefficients of the series obey the three-term recursion relations. To get the finite solution, one needs to truncate the series. It gives the spectrum ε_1 (34). If s is the power of the polynomial, then

$$\varepsilon_1 = 2s + \delta - \tfrac{1}{2}, \tag{37}$$

where δ was defined by (22).

The state can be labelled by indices k and q that are the numbers of positive and negative zeros of the partitial functions respectively. All roots of the polynomial are real, so $k + q = s$.

Transition to the strong field limit for the eigenvectors of $G_\lambda^{(\alpha)}$ operators can be done also in the basis of the eigenfunctions of the compact generators $M_3^{(\alpha)}$ with fixed parity. Introducing the scale-factor $\varkappa$ and the continuous variable $\mu = 2l\varkappa$ instead of the discrete index l in the TTRR (24) we arrive again at the equation (36).

5. Correlation diagram

The GCG states classification is analogous to that of the GC top and is based on the theorem that the tridiagonal Jacobi matrix $\langle n|G_\lambda^{(\alpha)}|n'\rangle$ has a non-degenerated real spectrum. If E, λ, b are fixed, the eigenvalues $g_q^{(2)}(E, \lambda, b)$ monotonously increase

$$g_q^{(2)} < g_{q+1}^{(2)}, \qquad q = 0, 1, 2 \ldots, \tag{38}$$

and eigenvalues $g_k^{(1)}(E, \lambda, b)$ monotonously decrease

$$g_k^{(1)} > g_{k+1}^{(1)}, \qquad k = 0, 1, 2, \ldots. \tag{39}$$

According to the Hellmann–Feynman theorem the derivatives of eigenvalues $g_n^{(\alpha)}(E, \lambda, b)$ with respect to energy at fixed λ and b are

$$\begin{aligned} \partial g_q^{(2)}/\partial E &= \langle g_q^{(2)}|-2u|g_q^{(2)}\rangle < 0, \\ \partial g_k^{(1)}/\partial E &= \langle g_k^{(1)}|-2v|g_k^{(1)}\rangle > 0. \end{aligned} \tag{40}$$

As a result of (38), (39) and (40) the following inequalities for $E_{kq}(\lambda, b)$ take place

$$\begin{aligned} E_{kq}(\lambda, b) &< E_{k+1\,q}(\lambda, b), \\ E_{kq}(\lambda, b) &< E_{k\,q+1}(\lambda, b). \end{aligned} \tag{41}$$

Therefore at fixed arbitrary λ and b and one of the quantum numbers, k or q, the eigenenergies monotonously increase, while another quantum number grows up. So k and q together with the parity δ can be called the true quantum numbers. If they are fixed, eigenvalues of integrals of motion do not appear and do not disappear as b changes. The sign of the derivatives $\partial g_{kq}/\partial\lambda$ is not fixed, but in the strong field limit it is minus according to (30).

To define a correlation diagram, it is sufficient to find the number of the state in the series $g_k^{(1)}|_{q=\text{fixed}}$ and $g_q^{(2)}|_{k=\text{fixed}}$, when $b=0$ and when $b\to\infty$. If one fixes the labels L and M, the dependence of E and g on λ becomes discontinuous, because the true indices k and q also become dependent on λ.

To find out the connections between L, M, λ and the true indices δ, k and q let us consider the polynomial

$$\tau_\lambda^{(\alpha)}(\mu) = \mu^3 - 4\lambda\mu^2 - \sigma\mu - g, \qquad \sigma = 2(E + \tfrac{1}{8} - 2\lambda^2), \tag{42}$$

which is the limiting form of the TTRR (24) at $b=0$. The negative and positive roots of the polynomial correspond to $\alpha = 1$ (variable v) and $\alpha = 2$ (variable u) respectively. Introducing $\tilde{n}_\alpha = L + (-1)^\alpha M + \frac{1}{2}$ one gets

$$\tau_\lambda^{(\alpha)}(\mu) = (\mu - \tilde{n}_2)(\mu + \tilde{n}_1)(\mu - 4\lambda - \tilde{n}_1 + \tilde{n}_2). \tag{43}$$

According to (24) the step on u and v variables in the $M_3^{(\alpha)}$ representation is equal to 2. To define the state number it is sufficient to count how many numbers $\mu_\alpha = (-1)^\alpha(2l + \delta + \frac{1}{2})$ satisfy the inequalities

$$\begin{aligned} \tau_\lambda^{(2)}(\mu) &< 0, \qquad \mu > 0, \\ \tau_\lambda^{(1)}(\mu) &> 0, \qquad \mu < 0. \end{aligned} \tag{44}$$

The relative position of the roots depends on λ and can be different. As a final result we have

$$M \geqslant 2\lambda$$

$$k = \begin{cases} \frac{1}{2}(3M - L - \delta) - [\frac{1}{2}(4\lambda + \frac{1}{2} + \delta)], & 4\lambda < 3M - L - \frac{1}{2}, \\ \frac{1}{2}(L - 3M + 2 - 3\delta) + [\frac{1}{2}(4\lambda + \frac{1}{2} + \delta)], & 4\lambda \geqslant 3M - L - \frac{1}{2}, \end{cases} \tag{45}$$

$$q = \tfrac{1}{2}(L + M + \delta),$$

$$M < 2\lambda$$

$$k = \tfrac{1}{2}(L - M - 3\delta + 2),$$

$$q = \begin{cases} \frac{1}{2}(L + 3M + \delta) - [\frac{1}{2}(4\lambda + \frac{3}{2} - \delta)], & 4\lambda < L + 3M + \frac{1}{2}, \\ [\frac{1}{2}(4\lambda - \frac{1}{2} + 3\delta)] - \frac{1}{2}(L + 3M + \delta), & 4\lambda \geqslant L + 3M + \frac{1}{2}. \end{cases} \tag{46}$$

Here brackets [ ] denote the integer part. The indices k and q have steps as λ changes. There are zones of λ, inside which, at fixed L and M, indices k and q are constant, when λ changes. The steps occur at the critical values $\lambda_c^{(\alpha)} = -(-1)^\alpha \frac{1}{8} - \frac{1}{4}\delta + \frac{1}{2}j_\alpha$, where j_α is an integer.

At the critical values of $\lambda_c^{(\alpha)}$ both the energy and the separation constant have a two-fold degeneration at $b = 0$. The labels of these two states (L, M) and (L', M') obey one of the conditions $L - M = L' - M'$ or $L + M = L' + M'$. In the first case $\lambda_c^{(1)} = \frac{1}{4}(L + M + 2M' + \frac{1}{2})$, in the second case $\lambda_c^{(2)} = \frac{1}{4}(2M' + M - L - \frac{1}{2})$. Two types of the critical λ correspond to the two types of permutations of the roots of the $\tau_\lambda^{(\alpha)}(\mu)$ polynomial, while the third root is fixed. Due to the obvious inequality $\mu_1 > \mu_2$ one should fix alternatively $\mu_1 = \tilde{n}_2$ or $\mu_2 = -\tilde{n}_1$.

In the strong field limit the energy of the asymptotic oscillator can be expressed in terms of L, M, λ. We have

$$M \geqslant 2\lambda$$

$$\varepsilon_1 = \begin{cases} 4M + \delta + \frac{1}{2} - 2[\frac{1}{2}(4\lambda + \frac{1}{2} + \delta)], & 4\lambda < 3M - L - \frac{1}{2}, \\ 2L - 2M + \frac{3}{2} - \delta + 2[\frac{1}{2}(4\lambda + \frac{1}{2} + \delta)], & 4\lambda \geqslant 3M - L - \frac{1}{2}, \end{cases} \tag{47}$$

$$M < 2\lambda$$

$$\varepsilon_1 = \begin{cases} 2L + 2M - \delta + \frac{3}{2} - 2[\frac{1}{2}(4\lambda + \frac{3}{2} - \delta)], & 4\lambda < 3M + L + \frac{1}{2}, \\ 2[\frac{1}{2}(4\lambda - \frac{1}{2} + 3\delta)] - 4M + \frac{3}{2} - 3\delta, & 4\lambda \geqslant 3M + L + \frac{1}{2}. \end{cases} \tag{48}$$

The energy dependence on λ also has the zone structure. For each GCG state (L, M) in the strong field limit the asymptotic oscillator energy ε_1 is constant in the range between the two neighbouring critical values of λ. The energy ε_1 is different in different zones.

It can be pointed out that the step functions appear also in the correlation diagrams of other quantum mechanical problems (see, for instance, the two Coulomb centres problem (Power 1973, Komarov *et al* 1976)).

The equations (47) and (48) demonstrate an interesting peculiarity of the asymptotic oscillator energy degeneration. The Hamiltonian (3) is invariant under the simultaneous inversion of the coordinate axes and λ. As a consequence in the strong field limit the GCG states form pairs of the multiplets of the same degenerating multiplicity at fixed λ and $M \geqslant 0$.

For the separation constant operator we also can see the zone structure of the spectrum. Inside the zone g_0 is a smooth function of λ and makes steps on the bounds of the zone.

Figures 4 and 5 demonstrate the zone behaviour of E and g as functions of b, λ for the 'ground' state ($L=0, M=0$).

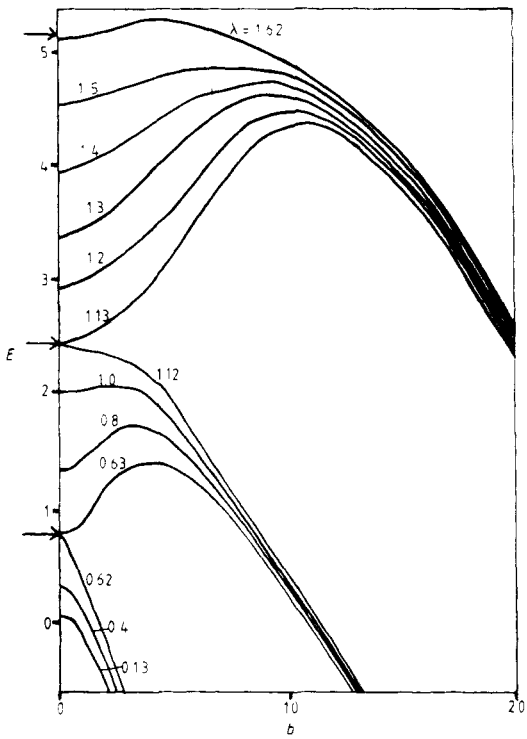

Figure 4. The energy dependence on field strength at various λ for the 'ground' state ($L=0$, $M=0$). The zone bounds are shown with arrows.

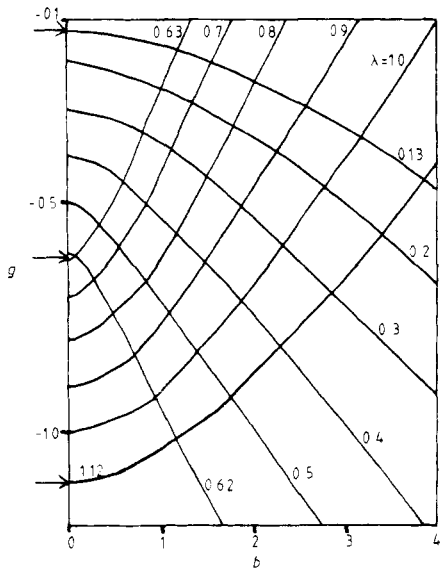

Figure 5. The separation constant dependence on field strengths at various λ for the 'ground' state ($L=0, M=0$). The zone bounds are shown with arrows.

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